Analysis of Longitudinal Thickness-Shear Stiffness of a Monolayer Filamentary Composite

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The longitudinal thickness-shear behavior of a rectangular cross-section prism containing circular cross-section fibers embedded in a matrix is analyzed. The problem is formulated as a Saint-Venant flexure problem modeled by a tip-loaded cantilever composite beam and solution is obtained by the boundary-point least-squares method. It is shown that the longitudinal thickness-shear modulus is much greater than the in-plane shear modulus for a monofilament composite and therefore an independent analysis is necessary to predict the longitudinal thickness-shear behavior of the composite. Numerical results are presented in graphical form for various values of fiber volume fraction and material parameters typical of modern advanced composites.

Nomenclature

A	= total cross-sectional area of a typical rectangular monofilament element
A_i	= cross-sectional area of a constituent region in a typical rectangular element; $i = f, m$
a_k	= coefficients of the fiber-region harmonic series solution
B	= constant defined in Eq. (34)
b_k, b_{-k}	= coefficients of the matrix-region harmonic series solution
C_1, C_2	= fiber-matrix interface and external boundaries, respectively
D_{11}, D_{12}, D_{22}	= longitudinal, Poisson and transverse flexural stiffnesses
D_{66}	= twisting stiffness
E	= Young's modulus
E_i	= Young's modulus of constituent material; $i = f, m$
E_{11}, E_{22}	= Young's moduli of a specially orthotropic material in x_j-directions ($j = 1, 2$)
G_i	= shear modulus of constituent material; $i = f, m$
G_{44}, G_{55}, G_{66}	= composite shear moduli: transverse thickness-shear, longitudinal thickness-shear, in-plane
I	= centroidal rectangular moment of inertia per unit length of a longitudinal cross section
I_x, I_y	= principal rectangular moments of inertia of a typical repeating cross section
I_E	= weighted moment of inertia defined in Eq. (35)
K	= shear coefficient
l	= length of cantilever
n	= outward normal coordinate on interface C_1
P'	= expression defined in the first of Eqs. (A5)
Q	= total shear force, Eq. (30)
Q_f, Q_m	= shear forces in the respective fiber and matrix regions
r	= radius of the fiber
S_{44}, S_{55}	= transverse and longitudinal thickness-shear stiffnesses
U, W	= mean displacements in x and z directions
u, v, w	= rectangular components of a displacement vector in directions x, y, z
u^i, v^i, w^i	= rectangular components of a displacement vector in the fiber ($i = f$) and matrix ($i = m$) regions
$\hat{u}, \hat{w}$	= residual displacements defined in Eqs. (3)
V	= volume
V_f, V_m	= fiber and matrix volume fractions, respectively
x, y, z	= rectangular coordinates; see Fig. 1a
x_1, x_2, x_3	= rectangular coordinates associated with the fiber (x_1), transverse (x_2), and thickness (x_3) directions
$\gamma_{avg}, \gamma_{eff}$	= average and effective thickness-shear strains
δ	= ratio of height to width of the typical composite cross section
ξ, η	= normalized rectangular coordinates
ρ, θ	= normalized polar coordinates
λ	= ratio of constituent-material shear moduli, $\lambda \equiv G_f/G_m$
λ_1	$= (\lambda - 1)/(\lambda + 1)$
μ	= ratio of width to fiber diameter of a typical monofilament cross section
v_f, v_m	= Poisson's ratios of the fiber and matrix materials
v_{ij}	= Poisson's ratios of an orthotropic material ($i, j = 1, 2, 3$)
$\bar{v}$	= mean value of Poisson's ratio
$\sigma_x, \sigma_y, \sigma_z$	= normal stresses in x, y, z directions
τ_{xz}, τ_{yz}	= longitudinal and transverse thickness-shear stresses, respectively
$\bar{\phi}$	= mean angle of rotation of the cross section in the xz plane
$\hat{\chi}^i, \chi^i$	= ordinary and normalized Saint-Venant flexure functions ($i = f, m$)
∇^2	= dimensionless Laplace operator; $\nabla^2 \equiv (\partial^2/\partial\xi^2) + (\partial^2/\partial\eta^2)$
$\hat{\nabla}^2$	= Laplace operator; $\hat{\nabla}^2 \equiv (\partial^2/\partial x^2) + (\partial^2/\partial y^2)$

Superscripts and Subscripts

$i = f, m$	= signifies the respective fiber ($i = f$) and matrix ($i = m$) materials
i, j, k, m, n	= dummy subscripts

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Introduction

WITH the development of advanced filamentary composite materials, modern structural designers are presented with a unique advantage over conventional monolithic materials. By laminating several layers of filamentary composites, with the filaments of each layer oriented in some prescribed direction, composite structures may be so designed as to give different properties in the prescribed directions as required by the particular application.

Owing to their oriented nonhomogeneous nature, one-layer filamentary composite materials behave as an orthotropic material on a macroscopic basis. Thus, to use these advantages fully, equivalent orthotropic properties of the composite must be characterized from the knowledge of constituent material properties and their respective geometrical configurations. In general, the complete characterization[1] of an orthotropic layer material requires: two moduli of elasticity E_{11}, E_{22}; one modulus of rigidity G_{66}; one independent Poisson's ratio ν_{12}; the longitudinal and transverse flexural stiffnesses D_{11} and D_{22}; flexural Poisson's stiffness D_{12}; twisting stiffness D_{66}; and the longitudinal and transverse thickness-shear stiffnesses‡ S_{55} and S_{44}. For most filamentary composite materials, the thickness-shear flexibility is expected to be significant[2,3] due to the presence of relatively flexible matrix material. This paper is concerned with an analysis to predict the longitudinal thickness-shear stiffness.

As the thickness-shear stresses are not distributed uniformly across the cross section, the effective thickness-shear strain needs to be known for the calculation of effective thickness-shear stiffnesses. It is a commonly accepted practice to relate the effective shear strain (γ_{eff}) to the average shear strain (γ_{avg}) by means of a correction factor K which is usually referred to as the shear coefficient[4]

$$K \equiv \gamma_{\text{avg}}/\gamma_{\text{eff}} \tag{1}$$

Since the thickness-shear strain distribution is dependent on the shape of the cross section on which the thickness-shear stress acts, K is also referred to as the shear shape factor.

In 1921, Timoshenko derived a theory of flexural beam vibration in which the effects of rotatory inertia as well as that of thickness shear were taken into account.[5] The shear coefficient K was defined as the ratio between the average shear strain and that at the midplane. This yielded a value of $\frac{2}{3}$ for a homogeneous, rectangular-section beam. However, this value of the shear coefficient gave poor agreement with experimental results. In view of this, numerous attempts were made to obtain a better value for K which would be in closer agreement with experimental results. Based on the high-frequency mode of beam vibration, Mindlin and Deresiewicz[6] obtained a value of 0.822 for a rectangular cross section; whereas, based on the static mode, Roark[7] gave a value of K of $\frac{5}{6}$. Recently, Cowper used a different static approach to derive a formula for K which is in good agreement with those obtained by other investigators. This latter approach is deemed to be satisfactory for long-wavelength low-frequency deformations such as those encountered in the vibration of monofilament composites.

Therefore, in the following, Cowper's analysis[8] is extended to the nonhomogeneous case consisting of a typical repeating element of a monofilament composite to obtain the effective thickness-shear strain. The thickness-shear stiffness, which is defined as the resultant shear force divided by the effective shear strain, can then be obtained as a direct consequence of the analysis.

Formulation

The usual micromechanics assumptions are made: 1) the fiber is of solid circular cross section, continuous, and of uniform diameter; 2) the fiber and matrix material are homogeneous, classical linear elastic, and may be transversely isotropic with the plane of isotropy normal to the fiber axis; 3) the bond along the entire fiber-matrix interface is perfect; and 4) the composite behaves macroscopically as a homogeneous orthotropic material.

First, the thickness-shear distribution is obtained from the analysis of a tip-loaded monofilament cantilever beam shown in Fig. 1a.

‡ Often referred to as transverse shear, here the term thickness shear is used, so that the term transverse can be reserved to refer to the direction normal to the longitudinal (filament) direction and contained in the plane of the layer.

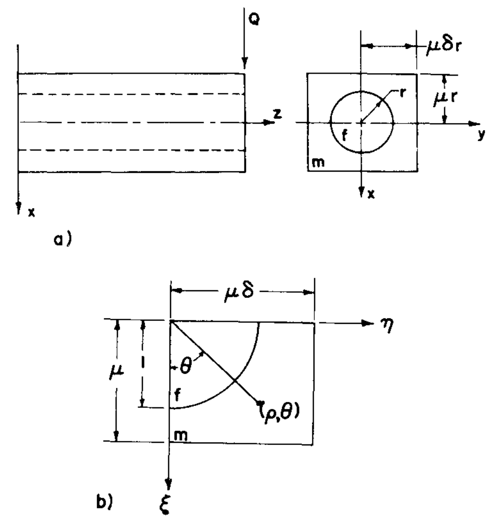

Fig. 1 Tip loading of a monofilament composite element; a) tip loading of a monofilament beam; b) cross-sectional view.

Denoting by u, v, w the displacement components in the x, y, z directions [Fig. 1a], one defines the mean displacements of the cross section in the x and z directions and the mean angle of rotation of the cross section in the xz plane by the following equations:

$$\left.\begin{aligned} U &= (1/A)\int\int_A u\,dx\,dy \\ W &= (1/A)\int\int_A w\,dx\,dy \\ \bar{\phi} &= (1/I_y)\int\int_A xw\,dx\,dy \end{aligned}\right\} \tag{2}$$

where A denotes the entire cross-sectional area, and I_y is the moment of inertia of the cross-sectional area with respect to the y axis. Then the actual displacements of a point on the cross section are written as

$$u = U + \hat{u}, \qquad w = W + x\bar{\phi} + \hat{w} \tag{3}$$

where $\hat{u}$ and $\hat{w}$ are the residual displacements which are equal to the deviations of the actual displacements from the weighted mean displacements. In view of the definitions, Eqs. (2)

$$\int\int_A \hat{u}\,dx\,dy = \int\int_A \hat{w}\,dx\,dy = \int\int_A x\hat{w}\,dx\,dy = 0 \tag{4}$$

The stress-strain relation in shear is

$$\tau_{xz}/G_i = u_{,z} + w_{,x} \tag{5}$$

where a comma represents partial differentiation with respect to the spatial variable that follows it, and the shear modulus G_i is to be interpreted as that of the fiber ($i = f$) or matrix ($i = m$) material depending on whether the material point is located in the fiber or the matrix region of the cross section. Equation (5) may be rewritten as follows:

$$W_{,z} + \bar{\phi} = (\tau_{xz}/G_i) - \hat{w}_{,x} - \hat{u}_{,z} \tag{6}$$

Finally, the integration of Eq. (6) yields the desired kinematic relations among the mean values of the midplane slope $W_{,z}$, flexural slope ϕ, and the effective shear strain γ_{eff}

$$W_{,z} + \bar{\phi} = (1/A)\int\int_A [(\tau_{xz}/G_i) - \hat{w}_{,x}]\,dx\,dy = \gamma_{\text{eff}} \tag{7}$$

For a tip loading, the shear force Q is uniform along the entire length of the beam, hence, the first part of the integral in Eq. (7) is evaluated as

$$\int\int_A (\tau_{xz}/G_i)\,dx\,dy = (Q_f/G_f) + (Q_m/G_m) \tag{8}$$

where Q_f and Q_m represent the respective shear forces that act on the fiber and matrix regions and are related to the total shear force Q by

$$Q = Q_f + Q_m \tag{9}$$

In view of Eq. (3), the remaining term in the integral of Eq. (7) is

$$\int\int_A \hat{w}_{,x}\,dx\,dy = \int\int_A (w_{,x} - \bar{\phi})\,dx\,dy \tag{10}$$

where $\bar{\phi}$ is defined in the third of Eqs. (2). Combining Eqs. (7–10), one obtains the effective thickness-shear strain expression

$$\gamma_{\text{eff}} = W_{,z}+\bar{\phi} = (1/A)[(Q_f/G_f)+(Q_m/G_m)- \int\int_A (w_{,x}-\bar{\phi})\,dx\,dy] \qquad (11)$$

where $\bar{\phi}$ and hence w, the displacement field, need to be known in terms of the parameters defining the geometrical configuration and the constituent properties before the integration can be carried out. Assuming that the deformation of the cross section can be approximated by that of a tip-loaded cantilever, and allowing the constituent materials to be transversely isotropic with the plane of isotropy normal to the fiber axis, one can readily formulate the problem using the Saint-Venant semi-inverse method.[9,10]

First, displacement components are assumed to be

$$\left.\begin{aligned} u^i &= B[\tfrac{1}{2}v_i(l-z)(x^2-y^2)+\tfrac{1}{2}lz^2-\tfrac{1}{6}z^3] \\ v^i &= Bv_i(l-z)xy \\ w^i &= -B\{x(lz-\tfrac{1}{2}z^2)+\hat{\chi}^i+[(E_i/G_i)-2v_i](\tfrac{1}{2}xy^2)\}\end{aligned}\right\}(i=f,m) \qquad (12)$$

where B is a constant to be determined from the boundary conditions, E_i is the Young's modulus in the fiber direction, G_i is the shear modulus in the vertical plane parallel to the fiber axis, v_i is to be interpreted as the Poisson's ratios $v_{31}=v_{32}$, and $\hat{\chi}^i=\hat{\chi}^i(x,y)$ $(i=f,m)$ are functions to be so chosen as to satisfy equilibrium conditions in their respective regions.

The stress components are readily calculated from the displacements, Eqs. (12), to be

$$\left.\begin{aligned} \sigma_x{}^i &= \sigma_y{}^i = \tau_{xy}{}^i = 0 \\ \tau_{xz}{}^i &= -BG_i\{\hat{\chi}_{,x}{}^i+(\tfrac{1}{2})v_1x^2+[(E_i/G_i)-3v_i](\tfrac{1}{2}y^2)\} \quad (i=f,m) \\ \tau_{yz}{}^i &= -BG_i\{\hat{\chi}_{,y}{}^i+[(E_i/G_i)-v_i]xy\} \\ \sigma_z{}^i &= -BE_i(l-z)x\end{aligned}\right\} \qquad (13)$$

If the constituent materials are isotropic, E_i, G_i, and v_i, respectively, of the constituent materials are related by

$$E_i = 2(1+v_i)G_i \qquad (i=f,m) \qquad (14)$$

Substitution of the stress components in Eqs. (13) into the equilibrium equation yields the following governing partial differential equations (Laplace's equations in two dimensions):

$$\hat{\nabla}^2(\hat{\chi}^i) = [(\partial^2/\partial x^2)+(\partial^2/\partial y^2)]\hat{\chi}^i = 0 \qquad (i=f,m) \qquad (15)$$

which must be satisfied in the respective regions A_f and A_m.

Consideration of displacement continuity at the fiber-matrix interface C_1 requires that

$$u^f = u^m, \quad v^f = v^m, \quad w^f = w^m \quad \text{on} \quad C_1 \qquad (16)$$

Apparently, the first and the second of Eqs. (16) cannot be satisfied unless the two Poisson's ratios are equal. However, the interaction between the constituent materials due to the differences in the Poisson's ratios has only weak effects as evidenced by many theoretical analyses; cf., Refs. 11–13. Hence, it will be assumed for the subsequent analysis that

$$v_f = v_m = \bar{v} \qquad (17)$$

With this assumption, Eq. (17), the first two of Eqs. (16) are identically satisfied and the third leads to

$$\hat{\chi}^f = \hat{\chi}^m \quad \text{on} \quad C_1 \qquad (18)$$

Continuity of surface traction at the interface C_1 requires that

$$\tau_{xz}{}^f(dx/dn)+\tau_{yz}{}^f(dy/dn) = \tau_{xz}{}^m(dx/dn)+\tau_{yz}{}^m(dy/dn) \qquad (19)$$

where n denotes the outward normal coordinate to C_1, hence, (dx/dn) and (dy/dn) are the direction cosines of the unit normal vector to the interface C_1.

In terms of the stress components defined in Eqs. (13), the condition of equilibrium at the interface, Eq. (19), is cast readily in the following form:

$$G_f(d\hat{\chi}^f/dn)-G_m(d\hat{\chi}^m/dn) = -(G_f-G_m)\{[\tfrac{1}{2}\bar{v}x^2+(1-\tfrac{1}{2}\bar{v})y^2](dx/dn)+(2+\bar{v})xy(dy/dn)\} \quad \text{on} \quad C_1 \qquad (20)$$

The condition that the lateral surface is free from surface traction leads to

$$(d\hat{\chi}^m/dn) = -\{[\tfrac{1}{2}\bar{v}x^2+(1-\tfrac{1}{2}\bar{v})y^2](dx/dn)+(2+\bar{v})xy(dy/dn)\} \quad \text{on} \quad C_2 \qquad (21)$$

The solutions to the problems posed by the governing differential Eqs. (15) and the boundary conditions, Eqs. (18, 20, and 21) are now obtained by assuming a pair of series solutions which are harmonic in the respective fiber and matrix regions. The coefficients of these series solutions are determined so as to satisfy the boundary conditions in the least-square-error sense.[14]

Solution

With the introduction of the following transformation

$$\left.\begin{aligned} &\xi = x/r, \quad \eta = y/r, \quad \chi^i = \hat{\chi}^i/r^3 \quad (i=f,m) \\ &\nabla^2 = (\partial^2/\partial\xi^2)+(\partial^2/\partial\eta^2)\end{aligned}\right\} \qquad (22)$$

Eqs. (15, 18, 20, and 21) defining the boundary-value problem, are rewritten in the following form, which is convenient for numerical analysis:

$$\nabla^2\chi^i = 0 \quad \text{in} \quad A_i (i=f,m) \qquad (23)$$

$$\chi^f = \chi^m \quad \text{on} \quad C_1 \qquad (24)$$

$$\lambda(d\chi^f/dn)-(d\chi^m/dn) = -(\lambda-1)\{[\tfrac{1}{2}\bar{v}\xi^2+(1-\tfrac{1}{2}\bar{v})\eta^2](d\xi/dn)+(2+\bar{v})\xi\eta(d\eta/dn)\} \quad \text{on} \quad C_1 \qquad (25)$$

$$(d\chi^m/dn) = -\{[\tfrac{1}{2}\bar{v}\xi^2+(1-\tfrac{1}{2}\bar{v})\eta^2](d\xi/dn)+(2+\bar{v})\xi\eta(d\eta/dn)\} \quad \text{on} \quad C_2 \qquad (26)$$

The series solutions, which are harmonic in the respective regions, are assumed to be

$$\left.\begin{aligned} \chi^f &= a_0 + \sum_{k=1}^{\infty} a_k\rho^k\cos k\theta \\ \chi^m &= b_0 + \sum_{k=1}^{\infty} (b_k\rho^k+b_{-k}\rho^{-k})\cos k\theta\end{aligned}\right\} \qquad (27)$$

In view of Eqs. (24) and (25), which warrant displacement continuity and stress equilibrium conditions, the coefficients a_0, a_k, and b_{-k} $(k=1,2,\ldots)$ may be expressed in terms of b_k as

$$\left.\begin{aligned} a_0 &= b_0 \\ a_1 &= 2b_1/(\lambda+1)-\lambda_1(3+2\bar{v})/4 \\ a_2 &= 2b_2/(\lambda+1)+(\lambda_1/4) \\ a_k &= 2b_k/(\lambda+1) \quad (k=3,4,\ldots) \\ b_{-1} &= -\lambda_1[b_1+(3+2\bar{v})/4] \\ b_{-2} &= -\lambda_1[b_2-(\lambda_1/4)] \\ b_{-k} &= -\lambda_1 b_k \quad (k=3,4,\ldots)\end{aligned}\right\} \qquad (28)$$

where

$$\lambda_1 \equiv (\lambda-1)/(\lambda+1) \qquad (29)$$

Because of the antisymmetry of the displacement component w with respect to the ξ axis, it may be shown readily that those coefficients with even subscripts are zero. The condition that the lateral surface is free from surface traction, Eq. (26), takes the form

$$\left.\begin{aligned} &\sum_{k=1,3,\ldots}^{\infty} kb_k[\rho^{k-1}\cos(k-1)\theta+\lambda_1\rho^{-k-1}\cos(k+1)\theta] = \\ &\quad -(\lambda_1/4)[(3+2\bar{v})\rho^{-2}\cos 2\theta-3\rho^{-4}\cos 4\theta]- \\ &\quad (\lambda-1)[\tfrac{1}{2}\bar{v}\xi^2+(1-\tfrac{1}{2}\bar{v})\eta^2] \\ &\qquad (0 \leqq \theta \leqq \tan^{-1}\delta, \quad \xi=\mu, \quad 0 \leqq \eta \leqq \mu\delta) \\ &\sum_{k=1,3,5}^{\infty} kb_k[-\rho^{k-1}\sin(k-1)\theta+\lambda_1\rho^{-k-1}\sin(k+1)\theta] = \\ &\quad -(\lambda_1/4)[(3+2\bar{v})\rho^{-2}\sin 2\theta+3\rho^{-4}\sin 4\theta]- \\ &\quad (\lambda-1)(2+\bar{v})\xi\eta \\ &\qquad (\tan^{-1}\delta \leqq \theta \leqq \pi/2, \quad 0 \leqq \xi \leqq \mu, \quad \eta=\mu\delta)\end{aligned}\right\} \qquad (30)$$

Next the coefficients b_k $(k=1,3,\ldots)$ are obtained according to the boundary-point least-square method.[14] With the coeffi-

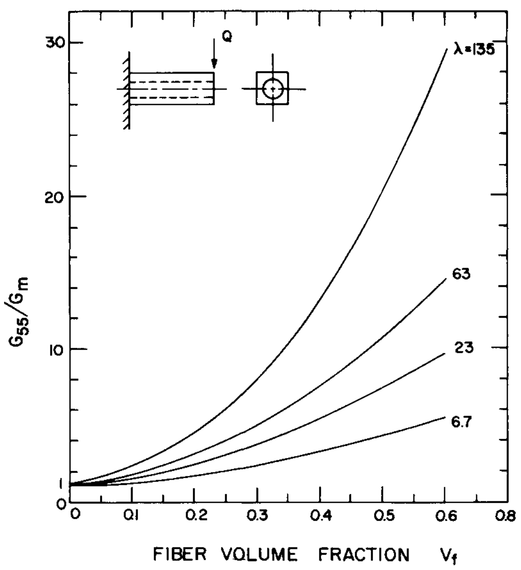

Fig. 2 Longitudinal thickness-shear modulus.

cients b_k thus obtained, the constant B which appears in Eqs. (12) and (13) is calculated from the condition that the resultant shear force is equal to the externally applied tip load Q

$$\int\int_A \tau_{xz}\,dx\,dy = Q \tag{31}$$

Since the stress components in Eqs. (13) satisfy equations of equilibrium in the absence of body forces, the following relation must hold

$$\sigma_{z,z}{}^i = BE_i x = -\tau_{zx,x}{}^i - \tau_{zy,y}{}^i \quad (i = f, m) \tag{32}$$

or

$$\tau_{zx,x}{}^i + \tau_{zy,y}{}^i + BE_i x = 0 \qquad (i = f, m) \tag{33}$$

In view of Eq. (33), Eq. (29) may be written as

$$\begin{aligned} Q &= \sum_{i=f,m} \int\int_{A_i} \tau_{xz}{}^i\,dx\,dy \\ &= \sum_{i=f,m} \int\int_{A_i} [(x\tau_{zx}{}^i)_{,x} + (y\tau_{zy}{}^i)_{,y}]\,dx\,dy \\ &\quad + B \sum_{i=f,m} E_i \int\int_{A_i} x^2\,dx\,dy \\ &= B \sum_{i=f,m} E_i \int\int_{A_i} x^2\,dx\,dy \\ &= BI_E \end{aligned} \tag{34}$$

where A_f and A_m are the cross-sectional areas of the respective fiber and matrix regions and I_E is the weighted moment of inertia of the cross section. From Eq. (33), the constant B is readily obtained as

$$B = Q/I_E \tag{35}$$

where

$$I_E \equiv \sum_{i=f,m} E_i \int\int_{A_i} x^2\,dx\,dy \tag{36}$$

Finally, the longitudinal thickness-shear stiffness S_{55} is obtained from Eq. (11) as

$$S_{55} = [Q/(2\mu\delta r)]/\gamma_{\text{eff}} = AI_E[\tfrac{1}{2}\bar{v}(I_x - I_y) - (A/I_y)\int\int_A x(\hat{\chi}^i + xy^2)\,dx\,dy]^{-1} \tag{37}$$

For an equivalent homogeneous beam with a rectangular cross section, the longitudinal thickness-shear stiffness is given by

$$S_{55} = KAG_{55}/(2\mu\delta r) \tag{38}$$

where the shear coefficient K is obtained as[8]

$$K = 10(1+\bar{v})/(12+11\bar{v}) \tag{39}$$

Thus, on equating Eqs. (37) and (38), one obtains the equivalent shear modulus G_{55} as

$$G_{55} = I_E(12+11\bar{v})[10(1+\bar{v})]^{-1}[\tfrac{1}{2}\bar{v}(I_x - I_y) - (A/I_y)\int\int_A x(\hat{\chi}^i + xy^2)\,dx\,dy]^{-1} \tag{40}$$

Numerical Results

The solution to the problem is obtained by assuming a twenty-term series solution for the normalized Saint-Venant flexural function in the matrix region. The twenty unknown coefficients b_k ($k = 1, 3, \ldots, 39$) are so determined that the boundary conditions at forty equally-spaced points on the boundary are satisfied in the least-square-error sense. The remaining coefficients, b_{-k} and a_k, defining the flexural functions χ^f and χ^m are readily related to the b_k's by Eq. (28). Several preliminary computer runs indicate that the series converges rapidly and the solution is relatively insensitive to the values of $\bar{v}$ ranging between 0.2 (boron) and 0.35 (epoxy). The dimensionless longitudinal thickness-shear modulus G_{55}/G_m is then readily calculated from Eq. (38) with the help of Eqs. (A5) and (A6) for various fiber-matrix material combinations and fiber volume fractions as summarized in Fig. 2.

For a small-fiber composite or parallel laminates consisting of many layers, it is generally accepted that the longitudinal thickness-shear modulus G_{55} is equal to the in-plane longitudinal shear modulus G_{66}.[15,16] However, in the case of a single-layer composite with only a single row of fibers, the longitudinal thickness-shear modulus G_{55} is expected to be greater than the in-plane shear modulus G_{66} as a result of the shear-stiffening effect of the fiber. Bert[17] used an approximate Jourawski-type shear theory to predict that the ratio G_{55}/G_{66} may vary from 2.86 for boron-aluminum to 37 for boron-epoxy composites with a volume fraction of 0.482. In the present more refined analysis, this ratio is found to vary from 1.5 for boron-aluminum to 16.2 for boron-epoxy composites with a volume fraction of 0.5.

Conclusions

The analysis presented here shows that the longitudinal thickness-shear modulus is much greater than the in-plane shear modulus for a single-row fiber composite layer. Thus, an independent analysis for the longitudinal thickness-shear modulus is necessary for complete characterization of the layer elastic properties. The numerical results for the longitudinal thickness-shear modulus are summarized graphically in Fig. 2 for the values of geometric and material parameters typical of advanced composites.

The stress distribution obtained from this analysis may be utilized to estimate the damping property of the composite associated with the longitudinal thickness-shear deformation by the approach reported previously in Ref. 18. Combining these predicted stiffness and damping coefficients values with those associated with the in-plane, flexural, twisting, and transverse-shear actions, one obtains a complex characterization of a single monofilament layer.[10] Such stiffness and damping properties have been used as input data for a forced vibration analysis of free-edge laminated plates and resulted in predictions of natural frequencies and associated nodal patterns and damping factors[19] which agreed well with experimental results reported by Clary.[20]

Appendix: Details of Solution

Saint-Venant Flexural Function $\chi^i(\rho, \theta)$ ($i = f, m$)

Flexural functions χ^f and χ^m that satisfy the Laplace equation in the respective fiber and matrix regions are assumed in series form as follows:

$$\left.\begin{aligned} \chi^f(\rho, \theta) &= \sum_{k=1,3}^{\infty} a_k \rho^k \cos k\theta \\ \chi^m(\rho, \theta) &= \sum_{k=1,3}^{\infty} (b_k \rho^k + b_{-k}\rho^{-k}) \cos k\theta \end{aligned}\right\} \tag{A1}$$

Note that in Eqs. (A1), the coefficients with even-numbered subscripts are zero due to the antisymmetry condition with respect to the η-axis.

Relationships among Fiber Region and Matrix-Region Coefficients, a_k and b_k

In view of the displacement-continuity and the stress-equilibrium requirements at the fiber-matrix interface, where $\rho = 1$, a_k are related to b_k, on substitution of Eq. (A1) into Eqs. (25) and (26), as follows:

$$\left.\begin{aligned} a_k &= b_k + b_{-k} \quad (k = 1, 3, \ldots) \\ \lambda a_k &= b_k - b_{-k} \quad (k = 5, 7, \ldots) \\ \lambda a_1 &= b_1 - b_{-1} - (\lambda - 1)(3 + 2\bar{\nu})/4 \\ \lambda a_3 &= b_3 - b_{-3} + (\lambda - 1)/4 \end{aligned}\right\} \quad \text{(A2)}$$

Equation (A2) may be solved for a_k and b_{-k} in terms of b_k to yield

$$\left.\begin{aligned} a_1 &= 2b_1/(\lambda+1) - \lambda_1(3+2\bar{\nu})/4 \\ a_3 &= 2b_3/(\lambda+1) + \lambda_1/4 \\ a_k &= 2b_k/(\lambda+1) \quad (k = 3, 5, \ldots) \\ b_{-1} &= -\lambda_1[b_1 + (3+2\bar{\nu})/4] \\ b_{-3} &= -\lambda_1[b_3 + (1/4)] \\ b_{-k} &= -\lambda_1 b_k \quad (k = 3, 5, \ldots) \end{aligned}\right\} \quad \text{(A3)}$$

where λ_1 is as defined in Eq. (29).

Longitudinal Thickness-shear Stiffness S_{55}

In view of notations in Eq. (22), the longitudinal thickness-shear stiffness expression, Eq. (37), may be written as

$$S_{55}/(E_m r) = 4\delta\mu^2[(\lambda' - 1)(\pi/4) + 4\delta\mu^4/3]/P' \quad \text{(A4)}$$

where

$$P' \equiv \tfrac{2}{3}\delta\mu^4[\bar{\nu}(\delta^2 - 1) - 2\delta^2] - (3/\mu^2)[(\pi a_1/4) + \textstyle\int\int_{A_m} \xi\chi^m \, d\xi \, d\eta]$$

$$\begin{aligned} \textstyle\int\int_{A_m} \xi\chi^m \, d\xi \, d\eta &= -(\pi b_1/4)[\mu^2 \tan^{-1}\delta - (\pi/4) + \mu^2\delta^2(\delta^{-1} - \tfrac{1}{2}\pi + \\ &\tan^{-1}\delta)] + b_{-3}[2(\tfrac{1}{4}\pi - \tan^{-1}\delta) + \tfrac{1}{4}\pi - 2\delta/(1+\delta^2)] + \\ &4\int_0^{\tan^{-1}\delta} G_1(\theta)\, d\theta + 4\int_{\tan^{-1}\delta}^{\pi/2} G_2(\theta)\, d\theta \end{aligned} \quad \text{(A5)}$$

$$\begin{aligned} G_1(\theta) \equiv{}& (b_1/4)\mu^4 \sec^2\theta + (b_3/6)(\mu\sec\theta)^6 \cos\theta\cos 3\theta + \\ &\sum_{k=5,7,\ldots}^{\infty} b_k[(k+3)^{-1}(\mu\sec\theta)^{k+3} - \lambda_1(-k+3)^{-1} \cdot \\ &(\mu\sec\theta)^{-k+3}] \cdot \cos\theta\cos k\theta \end{aligned}$$

$$\begin{aligned} G_2(\theta) \equiv{}& (b_1/4)(\mu\delta\csc\theta)^4 \cos^2\theta + (b_3/6)(\mu\delta\csc\theta)^6 \cos\theta\cos 3\theta + \\ &\sum_{k=5,7,\ldots}^{\infty} b_k[(k+3)^{-1}(\mu\delta\csc\theta)^{k+3} - \lambda_1(-k+3)^{-1} \cdot \\ &(\mu\delta\csc\theta)^{-k+3}]\cos\theta\cos k\theta \end{aligned}$$

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